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Failure waves in a prestressed layer of porous material☆

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Abstract

Unlike the traditional modelling of the reaction of high buildings to dynamic actions using standard software packages, which encounters fundamental difficulties due to the large number of structural components and the complex geometry of the joints, an alternative description of the collapse of a building is proposed in which the building is considered as a brittle continuum with a high initial porosity. To describe the accumulation of dispersed failures and the collapse of a building, an energy model of the damage of a brittle material with an initial porosity, which enables one to describe a number of qualitative features of its behaviour, is used. The most important of these features is the existence of a threshold deformation at which a damage accumulation begins, leading, in the final analysis, to the occurrence of rheological instability, which characterizes the tensile, shear and compressive strength of a porous medium. These limit states depend both on the parameters of the material as well as on the strain history. The propagation of stress waves caused by a pulsed compressive load applied to the upper boundary of a plane layer is investigated. It is shown that a relatively weak action can be transformed into a failure wave on account of the initial stresses in the material in a gravitational field. It is established that moderate compressive actions lead to shear fractures and that the stresses can both increase and decrease during the propagation of the head wave, depending on the level of the action. It is shown that intense actions lead to bulk failure which, due to strong damping, only occurs in the initial stage of the motion. The equation of a macrofailure wave is obtained. The conditions under which it is formed and the collapse retardation time for a dynamic action are investigated. It is shown that an increase in the shear modulus and in the absorption coefficient of the energy released increases the resistance of a building. A decrease in the parameters of the porous medium, which characterize the release of energy as a consequence of the damage evolution, will lead to the same thing. A new effect is revealed which is related to the weak dependence of a macrofailure wave on the applied stress.

Even a low intensity dynamic action on a high building can lead to its partial failure or total destruction due to the release of potential energy.[1] The use of difference schemes and finite-element methods to calculate dynamic processes is not very promising because of the accumulation of errors and loss of approximation due to the complex geometry of the joints between the structural components, the number of which is measured in thousands, and the need to carry out the calculations over long time intervals, which exceed the transit time of an acoustic wave through individual structural components by several order of magnitude.

An alternative approach to describing the failure of a high building is based on continuous, averaged models, according to which the building is treated as a continuous brittle body with an initial porosity. Such a material must possess a number of special behavioural features. These primarily relate to the elastic reaction at low stresses, the damage accumulation under intense stretching, shear and compression, the release of energy accompanying the failure of the prestressed medium, and the rapid growth and localization of damage in states which are close to macrofailure.

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A model of continuous failure,[2,3] based on the concept of an effective surface energy of an ensemble of microdefects and the latent energy of structural changes, is used to describe the material behaviour taking account of the above mentioned features. This energy approach is analogous to Griffith's approach in the mechanics of an isolated crack.[4,5] The central point of the description is that it takes account of the local balance between the elastic energy of a deformed body and the surface energy of numerous microfissures. An isotropic medium is considered, although the case of a material with orthotropic symmetry or transversal isotropy would be more realistic.

The kinetics of dispersed failure is based on the assumption that the damage rate is proportional to the derivative of the total energy, which is equal to the sum of the elastic potential and the surface energy, with respect to a damage parameter. The kinetic equation used not only ensures the non-negative dissipation in any processes of damage accumulation but also enables one to determine a natural scalar measure of the "dynamic overloading" of the material, which is applicable in any triaxial stress-strain state.

As in,[6,7] the Hadamard condition[8,9] of the reality of the velocities of the non-stationary characteristics of the dynamic equations is used as the strength criterion. Violation of this necessary condition of hyperbolicity leads to a loss of correctness of the boundary-value problems for the system of equations being considered, and means that it is impossible to describe the subsequent homogeneous deforming of the material within the framework of the model being studied. Hence, this constraint, which is imposed on the permissible strain and damage, plays the role of a strength criterion, which cannot be certainly exceeded. Unlike traditional strength criteria based on a constraint on the maximum principal stress, the forming energy, etc.,[10] the criterion used here is directly related to the deformation properties of the medium.

1. Basic relations

In the mathematical modelling of a continuum with a porosity coefficient close to unity, we will restrict ourselves, for simplicity, to the case of small strains and neglect the temperature effect. We shall assume that the medium is homogeneous and isotropic.

Suppose $\mathbf{u}$ is the displacement, $\mathbf{e} = 1/2(\nabla \otimes \mathbf{u} + (\nabla \otimes \mathbf{u})^T)$ is the symmetric tensor of small strains, and $\mathbf{v} = \dot{\mathbf{u}}$ is the mass velocity. Henceforth, a dot denotes a time derivative and ∇ is the gradient with respect to the spatial variable $\mathbf{x}$. By virtue of the smallness of the strains, the Lagrange and Euler variables are not distinguished. The sign $\otimes$ corresponds to external (tensor) multiplication.[11]

The actual state of a material is defined by the strains $\mathbf{e}$ and the scalar damage parameter $\omega \geq 0$. The reaction of the material is specified by the relations

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{e}, \omega), \quad u = u(\mathbf{e}, \omega), \quad u_f = u_f(\omega) \tag{1.1}$$

which define the symmetric stress tensor σ and the elastic potential u as isotropic functions of the strain and damage. The effective surface energy density u_f is assumed to be solely a function of the damage, since $\dot{u}_f = 0$ when $\dot{\omega} = 0$. The elastic potential and the effective surface energy (1.1) are given by the expressions[2,3]

$$\rho u(\mathbf{e}, \omega) = \frac{1}{2}KI_1^2(\mathbf{e}) + \mu J^2(\mathbf{e}) - \alpha_p(I_1)\omega I_1(\mathbf{e}) - \alpha_s \omega J(\mathbf{e}) \tag{1.2}$$

$$\rho u_f(\omega) = \rho u_f^0 + \gamma\omega + \frac{1}{2}\beta\omega^2 \tag{1.3}$$

Here ρ is the mass density, $I_1(\mathbf{e}) = \mathbf{e} : \mathbf{I}$, $J(\mathbf{e}) = (\mathbf{e}' : \mathbf{e}')^{1/2}$, and $\mathbf{e}' = \dot{\mathbf{e}} - \mathbf{I}_1\mathbf{I}/3$ is the deviator of the strains tensor. The material parameters K, μ, α_s, β, u_f^0, γ depend on the skeleton properties and the initial porosity.

The potential (1.2) is the expansion of a scalar isotropic function ρu ($\mathbf{e}$, ω) in a Taylor series up to terms of the second order of smallness with respect to the strain and the damage. The first two terms on the right-hand side of equality (1.2) correspond to a linearly elastic medium and the last two terms describe the change in the elastic energy as a consequence of damage accumulation. The minus sign in front of these two terms is associated with the reduction in energy as ω increases. The function $\alpha_p(I_1)$ is sign alternating-quantity which is negative under strong compression and positive under tension and moderate compression. The requirement of sign variability is due to the fact that the elastic energy must decrease as the damage builds up, which, in an initially porous, can develop both with stretching ($I_1(\mathbf{e}) > 0$) and with compression ($I_1(\mathbf{e}) < 0$).

The simplest method of determining the moduli K and μ, which depend on the initial porosity, is to use the moduli of the solid skeleton multiplied by its volume fraction. The method of asymptotic averaging of periodic structures[12,13] is particularly attractive.

The conservation laws for the medium being considered are represented by the equations

$$\dot{\mathbf{e}} = \frac{1}{2}(\nabla \otimes \mathbf{v} + (\nabla \otimes \mathbf{v})^T), \quad \rho\dot{\mathbf{v}} = \nabla \cdot \boldsymbol{\sigma}(\mathbf{e}, \omega) + \rho\mathbf{g} \tag{1.4}$$

$$\rho\dot{u}(\mathbf{e}, \omega) = \boldsymbol{\sigma} : \dot{\mathbf{e}} - \rho\dot{u}_f(\omega) - \delta \tag{1.5}$$

Distributed energy outflows are taken into account in Eq. (1.5): the quantity $\rho\dot{u}_f(\omega)$ determines the energy expenditure directly related to the increase in damage ω, the quantity δ in an isothermal process is equal to the heat removed (dissipation), i.e. the heat produced during the inelastic deformation of the material. A dot between vectors (tensors) denotes a scalar product and a colon denotes a double scalar product such that $\mathbf{A} : \mathbf{B} = A_{ij}B^{ij}$.

The kinetic equation for the damage is taken in the form

$$\dot{\omega} = \frac{1}{\tau\beta}\left\langle -\frac{\partial\rho(u + u_f)}{\partial\omega}\right\rangle = \frac{1}{\tau\beta}\langle\alpha_p(I_1)I_1 + \alpha_s J - \gamma - \beta\omega\rangle \tag{1.6}$$

where $\tau = \text{const} > 0$ is the stress relaxation time due to the increase in damage and $\langle x\rangle = 1/2(x + |x|)$. Relation (1.6) expresses the assumption that the damage rate is significantly associated with the balance between the elastic and surface energies. An increase in the damage only occurs when the energy release rate ($\partial\rho u/\partial\omega < 0$) is greater in absolute magnitude than absorption rate its ($\partial\rho u_f/\partial\omega > 0$). This leads to the inequality

$$\alpha_p(I_1(\mathbf{e}))I_1(\mathbf{e}) + \alpha_s J(\mathbf{e}) - \gamma - \beta\omega > 0, \quad \omega \geq 0 \tag{1.7}$$

which is the condition for the process to be active, that is, $\dot{\omega} > 0$. If this condition is not satisfied, the damage remains unchanged. It follows from Eq. (1.6) that, in any active deformation process, the dissipation

$$\delta = -\frac{\partial\rho(u + u_f)}{\partial\omega}\dot{\omega} = \frac{1}{\tau\beta}\left[\frac{\partial\rho(u(\mathbf{e}, \omega) + u_f(\omega))}{\partial\omega}\right]^2 \geq 0$$

The quantity $-\rho\partial(u + u_f)/\partial\omega$, which is equal to the rate of change of the sum of the elastic and surface energies with respect to an increment in the damage, is a natural scalar measure of the “dynamic overloading”,[14] which is applicable for an arbitrary tri-axial stress-strain state. Equation (1.6) enables one to describe the qualitative features of dispersed failure of porous media uniformly, in particular, the existence of threshold values of the strain at which the damage begins, the possibility of failure under stretching, shear and compression and the absence of a damage evolution in the case of unloading below the threshold. In a slow process ($\dot{\omega} \to 0$), relation (1.6) leads to a damage model with “instant kinetics”[3] when the local balance of the elastic and surface energy gives the finite relation between the damage and the actual strain

$$\omega = \beta^{-1}[\alpha_p(I_1(\mathbf{e}))I_1(\mathbf{e}) + \alpha_s J(\mathbf{e}) - \gamma]$$

At fixed damage, inequality (1.7) leads to the subdivision of the half-plane ($I_1, J \geq 0$) of the invariants of the strain tensor into two domains (Fig. 1). Putting $\omega = 0$ in condition (1.7), we arrive at the curve

$$J = f(I_1) = [\gamma - \alpha_p(I_1)I_1]/\alpha_s \tag{1.8}$$

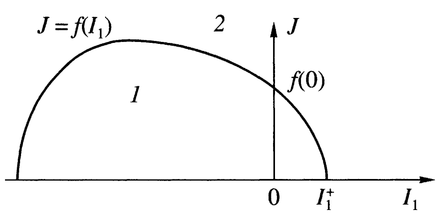

Fig. 1.

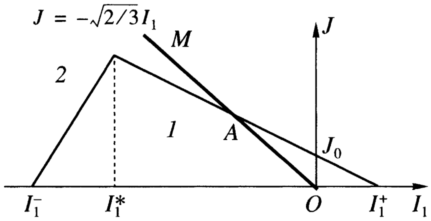

Fig. 2.

which bounds domain *1* of the undamaged elastic material. This curve possesses considerable asymmetry with respect to the *J* axis, which is due to the considerable difference in the tensile and compressive strengths. Domain *2* corresponds to the damaged material. If the boundary $J=f(I_1)$ is known, then a relation between the function $\alpha_p(I_1)$ and the equation of the boundary of the elastic domain follows from equality (1.8)[3].

The piecewise-linear approximation of this boundary[2,3]

$$J(I_1) = \begin{cases} J_0(1 - I_1/I_1^+), & I_1^* \le I_1 \le I_1^+ \\ J_1(1 - I_1/I_1^-), & I_1^- \le I_1 \le I_1^* \end{cases}, \quad J_1 \equiv J_0 \frac{1 - I_1^*/I_1^+}{1 - I_1^*/I_1^-}$$

shown in Fig. 2 is used below. This piecewise-linear approximation of the boundary of the elastic domain enables one to simplify the calculations considerably while, at the same time, preserving all the qualitative features of the behaviour of a medium with an initial porosity. In this case, the dependence is a piecewise-constant, sign-variable function.

$$\alpha_p(I_1) = \begin{cases} \alpha_p^+ \equiv \alpha_s J_0/I_1^+ > 0, & I_1^* \le I_1 \le I_1^+ \\ \alpha_p^- \equiv \alpha_s J_1/I_1^- < 0, & I_1^- \le I_1 \le I_1^* \end{cases}$$

It follows from Eq. (1.5) that the stress tensor is related to the elastic potential as follows:

$$\boldsymbol{\sigma} = \rho\frac{\partial u(\mathbf{e}, \omega)}{\partial \mathbf{e}} = (KI_1(\mathbf{e}) - \alpha_p(I_1)\omega)\mathbf{I} + \left(2\mu - \frac{\alpha_s\omega}{J(\mathbf{e})}\right)\mathbf{e}' \tag{1.9}$$

Relation (1.9) between the stresses and the actual strain and damage shows that the tangential moduli of the material being considered may fall to zero as the damage builds up. This means that, in strain space, the permissible states must be bounded by a certain limit surface (a strength criterion).

2. Characteristics and strength criterion of a damaged material

In order to formulate a strength criterion for the material, we will use the Hadamard condition,[8,9] that is, the condition for which the velocities with which the non-steady-state characteristics of the dynamic equations propagate are real. A violation of the necessary condition of the hyperbolicity of the dynamic system of equations (1.4) and (1.6) signifies a loss of correctness of the boundary-value problem as a consequence of which further homogeneous deformation of the material cannot occur within the framework of the model being studied. Since the damage evolution is determined by the kinetic equation (1.6), the limit surface depends not only on the actual strains but also on their history. At the same time, this strength criterion is directly related to the mechanical properties of the material, which are determined by the elastic potential (1.2) and the effective surface energy (1.3).

The dynamics of the initially porous material which is being damaged in the active process ($\dot{\omega} > 0$) is described by the system of equations (1.4), (1.6) and (1.9). Suppose $\mathbf{N}=\mathbf{e}'/J(\mathbf{e})$ is a normalized strain tensor and that c and $\mathbf{n}$ are the propagation velocity and the normal to the surface of a weak discontinuity. The non-stationary characteristics ($c \neq 0$) are defined by the equations

$$\begin{gathered} \det(\rho c^2\mathbf{I} - \mathbf{A}) = 0, \quad \mathbf{A}(\mathbf{e}, \omega, \mathbf{n}) = M\mathbf{I} + \Lambda\mathbf{n}\otimes\mathbf{n} + \xi\mathbf{m}\otimes\mathbf{m} \\ \xi(\mathbf{e}, \omega) = \alpha_s\omega/J(\mathbf{e}), \quad M(\xi) = \mu - \xi/2, \quad \Lambda(\xi) = \lambda + \mu - \xi/6, \quad \mathbf{m} = \mathbf{N}(\mathbf{e})\cdot\mathbf{n} \end{gathered} \tag{2.1}$$

The eigenvalues of the acoustic tensor A are given by the expressions

$$\rho c_{1,2}^2(\mathbf{e}, \xi, \mathbf{n}) = M + P \pm (P^2 - Q)^{1/2}, \quad \rho c_3^2(\mathbf{e}, \xi) = M \tag{2.2}$$

$$\begin{aligned} 2P(\mathbf{e}, \xi, \mathbf{n}) &= \Lambda + \xi\mathbf{m}\cdot\mathbf{m} = \lambda + \mu + \xi(\mathbf{n}\cdot\mathbf{N}^2(\mathbf{e})\mathbf{n} - 1/6) \\ Q(\mathbf{e}, \xi, \mathbf{n}) &= \xi\Lambda(\mathbf{m}\cdot\mathbf{m} - (\mathbf{m}\cdot\mathbf{n})^2) = \xi\Lambda(\mathbf{n}\cdot\mathbf{N}^2(\mathbf{e})\cdot\mathbf{n} - (\mathbf{n}\cdot\mathbf{N}(\mathbf{e})\cdot\mathbf{n})^2) \end{aligned} \tag{2.3}$$

From relations (2.2) and (2.3) it can be seen that a direction $\mathbf{n}$ and a value of ξ exist for which the velocity of the non-stationary characteristics tends to zero. In this case, the positive definite acoustic tensor $\mathbf{A}$ is degenerate. We shall say[6,7] that a state $(\mathbf{e}^*, \xi^*)$ of a material element is rheologically unstable if a direction $\mathbf{n}_*(\mathbf{e}^*, \xi^*)$ exists such that $c(\mathbf{e}^*, \xi^*, \mathbf{n}_*) = 0$.

Rheological instability can be accompanied by the formation of surfaces of a strain localization with a certain orientation $\mathbf{n}_*(\mathbf{e}^*(\mathbf{x}, t), \omega^*(\mathbf{x}, t))$ of these elements. Actually, the vector for the discontinuity in the normal derivative of the mass velocity $\mathbf{V} = [\partial\mathbf{v}/\partial n]$, which is given by the equation $(\rho c^2\mathbf{I} - \mathbf{A})\,\mathbf{V} = 0$ when $c \to 0$, can be non-vanishing. It then follows from the first equation of system (1.4) that, when $\mathbf{V} \neq 0$, the discontinuity in the normal derivative of the strain tensor $\mathbf{E} = [\partial\mathbf{e}/\partial n]$ increases without limit as $c \to 0$. This means that a weak discontinuity becomes a strong discontinuity, on which the velocity is continuous, but the strain undergoes a discontinuity. Such a discontinuity is the surface of a strain localization.

It follows from relations (2.2) that degeneracy with respect to the velocity $\rho c_3^2(\mathbf{e}, \omega)$ occurs in the states

$$\xi^* = 2\mu \tag{2.4}$$

The orientation of such strain localization surfaces is arbitrary.

The extremum of the velocity $\rho c_2^2(\mathbf{e}, \xi, \mathbf{n})$ is defined by the equation

$$(\mathbf{I} - \mathbf{n}\otimes\mathbf{n})\cdot\mathbf{B}\cdot\mathbf{n} = 0; \quad \mathbf{B} \equiv (\Lambda + M - \rho c_2^2)\mathbf{N}^2(\mathbf{e}) - 2\Lambda(\mathbf{n}\cdot\mathbf{N}(\mathbf{e})\cdot\mathbf{n})\mathbf{N}(\mathbf{e}) \tag{2.5}$$

It is clear that $\mathbf{n}$ is an eigenvector of the symmetric tensor $\mathbf{B}$, which is a quadratic polynomial of the symmetric tensor $\mathbf{N}(\mathbf{e})$ with coefficients which depend on the parameter ξ and the invariants $\mathbf{n}\cdot\mathbf{N}(\mathbf{e})\mathbf{n}$, $\mathbf{n}\cdot\mathbf{N}^2(\mathbf{e})\mathbf{n}$. The eigenvectors of the tensor $\mathbf{N}$ are therefore eigenvectors of the tensor $\mathbf{B}$. The converse, generally speaking, is not true.

It can be shown that, when $c_2 = 0$, system (2.5) has the following solutions:

1) the normal $\mathbf{n}$ coincides with one of the eigenvectors of the tensor $\mathbf{N}(\mathbf{e})$; loss of rheological stability of the material occurs when $\xi^* = 2\mu$, the surfaces on which the strain is localized are oriented perpendicular to the principal axes of the tensor $\mathbf{N}(\mathbf{e})$;
2) if two eigenvalues of the tensor $\mathbf{B}$ are identical (to be specific, $B_1 = B_2$), then rheological instability arises when $\xi^* > 2\mu$, where ξ^* is a root of the equation

$$\begin{aligned} &M + P - (P^2 - Q)^{1/2} = 0 \\ &2P = \Lambda + \xi^*(N_1^2 n_1^2 + N_2^2 n_2^2), \quad Q = \xi^*\Lambda(N_1 - N_2)^2 n_1^2 n_2^2 > 0 \\ &M(\xi^*) = \mu - \frac{1}{2}\xi^*, \quad \Lambda(\xi^*) = \lambda + \mu - \frac{1}{6}\xi^* \end{aligned} \tag{2.6}$$

and the planes of discontinuity pass through one of the principal axes of the tensor $\mathbf{N}(\mathbf{e})$ and make an angle between the two other principal axes in accordance with the formulae (for any cyclic permutation of 1, 2 and 3):

$$2n_1^2 = (1 - M\Lambda^{-1}N_3/(N_1 - N_2)), \quad n_2^2 = 1 - n_1^2, \quad n_3 = 0 \tag{2.7}$$

3) if the tensor $\mathbf{B}$ is a spherical tensor, that is only possible in the case of uniaxial strain (for any cyclic permutation of 1, 2 and 3)

$$\mathbf{e} = e(t)\mathbf{e}_1\otimes\mathbf{e}_1, \quad \mathbf{N}(\mathbf{e}) = \kappa\sqrt{2/3}\mathbf{e}_1\otimes\mathbf{e}_1 - \kappa(\mathbf{e}_2\otimes\mathbf{e}_2 + \mathbf{e}_3\otimes\mathbf{e}_3)/\sqrt{6}, \quad \kappa = \operatorname{sign} e(t)$$

then the extremal normal coincides with the normal to the surface of a circular cone with axis $\mathbf{e}_1$. The semi-aperture angle Ψ of this cone is given by the expression

$$n_1^2 \equiv \sin^2\Psi = 1/2(1 + 1/3M\Lambda^{-1}), \quad n_2^2 + n_3^2 = 1 - n_1^2 \tag{2.8}$$

The value of the parameter $\xi^* > 2\mu$ is given by Eq. (2.6) for the normal (2.8).

We shall assume that a form of rheological instability first arises during the loading process, which corresponds to the smallest value of the instant of time t_* when the quantity $\xi(\mathbf{e}(t_*), \omega(t_*))$, for a specified strain trajectory, becomes equal to the critical value ξ^*. At this instant of time, one of the sound velocities vanishes and, then, $t_* = \min t^{(i)}$, where the values of $t^{(i)}$ are given by the equation

$$\alpha_s\hat{\omega}[\mathbf{e}(s), t^{(i)}]_{s=0}^{s=t^{(i)}} = \xi^* J(\mathbf{e}(t^{(i)})) \tag{2.9}$$

where $\hat{\omega}$ is the solution, which depends parametrically on t, of the kinetic equation (1.6) with zero initial conditions and the parameter ξ^* is either equal to 2μ or to a root of Eq. (2.6).

3. The head stress wave

We will now consider the propagation of the wave caused by a normal compressive stress $\sigma_0 = -p_0 H(t)$, $p_0 = \text{const} > 0$ applied to the upper boundary $x^1 = 0$ of a pre stressed layer $0 \le x^1 \le h$ of a highly porous material. Here, $H(t)$ is the Heaviside function, the $x^1 \equiv x$ axis of the Cartesian coordinates x^k with an orthonormal basis $\mathbf{e}^k$ $(k = 1, 2, 3)$ is directed along the normal to the boundary, and the x^2 and x^3 axes belong to the boundary $x^1 = 0$. The damage and mass velocity are equal to zero in the initial equilibrium state and the strain and stress have the form

$$\begin{aligned} &\mathbf{e}^0 = e^0\mathbf{e}_1 \otimes \mathbf{e}_1, \quad \boldsymbol{\sigma}^0 = \sigma_{11}^0\mathbf{e}_1 \otimes \mathbf{e}_1 + \sigma_{22}^0(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3) \\ &\sigma_{11}^0(x) = -\rho g x, \quad \sigma_{22}^0(x) = -\lambda g x/\Lambda_0, \quad e^0(x) = -gx/\Lambda_0; \quad \Lambda_0 = \lambda + 2\mu \end{aligned} \tag{3.1}$$

where g is the acceleration due to gravity directed along the x axis, λ and μ are the elastic moduli of the porous material, where the material parameters and the acceleration due to gravity satisfy the inequality

$$gh/\Lambda_0 < \gamma/\left|\alpha_p^+ + \alpha_s\sqrt{2/3}\right|$$

which means that there is no damage in the initial state. During the loading process, the strain tensor remains a uniaxial compression tensor $\mathbf{e} = e(x, t)\mathbf{e}_1 \otimes \mathbf{e}_1$, $e < 0$ and the invariants and strain deviator are given by the formulae

$$I_1(\mathbf{e}) = e, \quad J(\mathbf{e}) = -\sqrt{2/3}e, \quad \mathbf{e}' = e(2/3\mathbf{e}_1 \otimes \mathbf{e}_1 - 1/3(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3))$$

The straight line OAM in the half-plane (I_1, J) corresponds to this strain for which $J = -\sqrt{2/3}I_1$ (Fig. 2). Depending on the values of the material parameters, one of two possible modes of deformation and failure of the material occurs. In the first case, when the straight line OAM intersects the right-hand side of the boundary of the domain of the undamaged material, i.e.

$$J_0(1 - I_1^*/I_1^+) \le -\sqrt{2/3}I_1^* \tag{3.2}$$

the damage begins at moderate compression $I_1^* < I_1 < 0$ when a shear mechanism for the failure of the material predominates. Then, as $|I_1|$ increases, the strain state passes into the domain of strong compression $I_1^- < I_1 < I_1^*$. In the second case, the boundary of the elastic domain is such that the straight line OAM of uniaxial compression at once intersects the left-hand side of the boundary. Here, the damage begins at once by a bulk failure mechanism

We shall assume that the first form of loading occurs when the straight line $J = -\sqrt{2/3}\,I_1$ in Fig. 2 intersects the right-hand side of the boundary of the elastic domain. In this case, inequality (3.2) holds or, what is the same thing,

$$\alpha_p^+ I_1^+ \le (\alpha_p^+ - \alpha_s)\sqrt{2/3}I_1^*$$

The uniaxial dynamic compression of the material being considered is then described by a system of three differential equations in the unknowns $(\upsilon_1,\ \sigma_{11},\ \omega)$

$$\rho\frac{\partial\upsilon_1}{\partial t}-\frac{\partial\sigma_{11}}{\partial x} = \rho g,\quad \frac{\partial\sigma_{11}}{\partial t}-\Lambda_0\frac{\partial\upsilon_1}{\partial x} = -\frac{\alpha_\pm}{\tau}z_\pm(\sigma_{11},\omega),\quad \frac{\partial\omega}{\partial t} = \frac{1}{\tau}z_\pm(\sigma_{11},\omega)$$
$$\sigma_{11} = \Lambda_0 e-\alpha_\pm\omega,\quad z_\pm(\sigma_{11},\omega)\equiv\frac{\alpha_\pm\sigma_{11}-\gamma\Lambda_0}{\beta\Lambda_0}-\left(1-\frac{\alpha_\pm^2}{\beta\Lambda_0}\right)\omega,\quad \alpha_\pm = \alpha_p^\pm-\alpha_s\sqrt{2/3} \tag{3.3}$$

The plus and minus signs correspond to the intervals $I_1^* < I_1 < 0$ and $I_1 < I_1^*$.

In the dimensionless variables

$$\bar{t} = \frac{t}{t_0},\quad \bar{x} = \frac{x}{h},\quad \bar{\upsilon} = \frac{\upsilon_1}{c_0},\quad \bar{\sigma} = \frac{\sigma_{11}}{\Lambda_0},\quad \bar{\sigma}_0 = \frac{\sigma_0}{\Lambda_0},\quad \bar{g} = \frac{\rho g h}{\Lambda_0},\quad \bar{\tau} = \frac{\tau}{t_0}$$
$$\bar{\lambda} = \frac{\lambda}{\Lambda_0},\quad \bar{\mu} = \frac{\mu}{\Lambda_0},\quad \bar{\alpha}_\pm = \frac{\alpha_\pm}{\Lambda_0},\quad \bar{\beta} = \frac{\beta}{\Lambda_0},\quad \bar{\gamma} = \frac{\gamma}{\Lambda_0};\quad c_0 = \left(\frac{\Lambda_0}{\rho}\right)^{1/2},\quad t_0 = \frac{h}{c_0}$$

system (3.3) can be written in the form

$$\frac{\partial\upsilon}{\partial t}-\frac{\partial\sigma}{\partial x} = g,\ \frac{\partial\sigma}{\partial t}-\frac{\partial\upsilon}{\partial x} = -\frac{\alpha_\pm}{\tau}z_\pm,\ \frac{\partial\omega}{\partial t} = \frac{1}{\tau}z_\pm;\ z_\pm(\sigma,\omega)\equiv\frac{\alpha_\pm\sigma-\gamma}{\beta}-b_\pm^2\omega,\ b_\pm^2 = 1-\frac{\alpha_\pm^2}{\beta} \tag{3.4}$$

Henceforth, the bar over dimensionless variables is omitted.

At an infinitesimal strain rate, which corresponds to $\dot{\omega}\to 0$, the right-hand side of the last equation of system (3.4) also tends to zero. In this case, the damage ω can be expressed in explicit form in terms of the strain e or the stress σ:

$$\omega = \beta^{-1}(\alpha_\pm e-\gamma),\quad \beta\Lambda_0\omega-\alpha_\pm^2\omega = \alpha_\pm\sigma_{11}-\gamma\Lambda_0$$

It can be seen from this that $\alpha_\pm = \alpha_\pm - \alpha_s\sqrt{2/3} < 0$ since, in the case of compression $e<0$ and damage $\omega>0$. The expression for the component of the stress tensor σ_{11} at an infinitesimal strain rate reduces to the piecewise-linear dependence

$$\sigma_{11}(e) = \begin{cases}\Lambda_0 e, & e_A<e<0\\ \Lambda_f^+(e-e_A)+\sigma_A, & e_B<e<e_A\\ \Lambda_f^-(e-e_B)+\sigma_C, & e<e_B\end{cases}$$

Henceforth,

$$e_A = \gamma/\alpha_+,\quad \sigma_A = \Lambda_0 e_A = \Lambda_0\gamma/\alpha_+,\quad e_B = I_1^*,\quad \sigma_C = \Lambda_f^- e_B+\gamma\alpha_-/\beta,\quad \Lambda_f^\pm = \Lambda_0-\alpha_\pm^2/\beta$$

The relation $\sigma_{11}(e)/\Lambda_0$ is represented in Fig. 3 by the broken line *OABCD* for a material with the parameters

$$\lambda = \mu = 0.33,\quad \alpha_p^+ = 0.3,\quad \alpha_{p-} = -0.02,\quad \alpha_s = 1.0,\quad \gamma = 0.01,\quad \beta = 1.0$$

At the point $e_B = I_1^*$, this relation undergoes a discontinuity. Using the relations

$$\sigma_B = \Lambda_f^+(e_B-e_A)+\sigma_A,\quad \sigma_C = \Lambda_f^- e_B+\gamma\alpha_-/\beta$$

we obtain an expression for the discontinuity in the stress

$$\sigma_C-\sigma_B = \beta^{-1}(\alpha_+-\alpha_-)\{\alpha_+(e_B-e_A)+\alpha_- e_B\}>0$$

The non-negativity of this discontinuity follows from the compression condition $e<0$ and the inequalities

$$\alpha_+>\alpha_-,\quad \alpha_+<0,\quad \alpha_-<0$$

which follow from the definition of α_+.

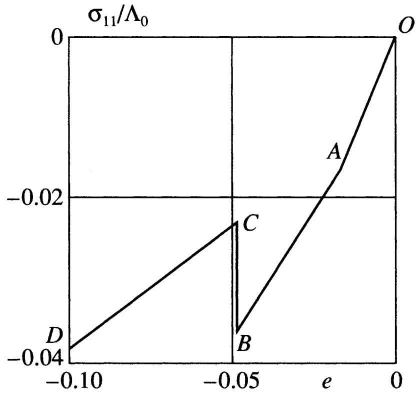

Fig. 3.

Hence, in the case of static strain, the attainment of point B, which corresponds to the volume compression I_1^*, is accompanied by a break in the compressive stress, that is, the static diagram of the material has a falling segment. However, due to dependence of the stresses on the strain rate, instability phenomena associated with a loss of hyperbolicity do not arise here. The slope of the segment CD is always less than the slope of the segment AB by virtue of the inequality $\alpha_- < \alpha_+ < 0$.

The boundary conditions for system (3.4) have the form

$$\sigma(x,0) = -gx, \quad \upsilon(x,0) = \omega(x,0) = 0, \quad 0 \le x \le 1 \tag{3.5}$$

$$\sigma(0,t) = \sigma_0, \quad \upsilon(1,t) = 0, \quad t \ge 0 \tag{3.6}$$

In the case of the quasilinear hyperbolic system (3.4), the velocities of the characteristics $c = \pm 1$. The solution of the problem contains a head shock wave with its origin at the point $x = t = 0$ related to the discontinuity of the boundary conditions (3.5) and (3.6) at this point. We shall consider system (3.4) in the triangle $\{0 \le x \le t, t \ge 0\}$ which corresponds to the first traverse of the shock wave through the prestressed layer. The boundary conditions are written in the form

$$\sigma(0,t) = \sigma_0, \quad \upsilon(t,t) = -(gx(t) + \sigma(t,t)), \quad \omega(t,t) = 0, \quad 0 \le t \le 1 \tag{3.7}$$

and are the first relation of (3.6) and the condition at the strong discontinuity $x = t$, written taking account of the initial state (3.5) in the unperturbed domain $\{x < t, t > 0\}$.

We will now consider the evolution of the stresses $\sigma_*(t) \equiv \sigma(t-0, t)$ on the shock wave front. Here, three different cases are possible: high, moderate and low pressure applied to the upper boundary $x = 0$.

In the case of a *high pressure* ($\sigma_0 < \sigma_B$, $\sigma_B = e_B$), the shock compression of a material with an initial porosity causes a volume deformation which exceeds (in modulus) the value $e_B \equiv I_1^*$ which separates the domains of shear and bulk failure. In this case, the coefficient $\alpha_\pm$ is selected with the plus sign which corresponds to a strain $e < I_1^*$. The right-hand side $(\alpha_-\sigma - \gamma)/\beta$ of the kinetic equation at the point $x = t = 0$ is greater than zero and a damage therefore already starts at the instant the load is applied. The propagation of a head wave is accompanied by a reduction in its amplitude $\sigma_*(t)$. At the instant of time t_B, such that $\sigma_*(t_B) = \sigma_B$, the character of the dispersed failure changes and bulk failure is replaced by shear failure. The change in the stresses in the shock wave is given by the Cauchy problem

$$\begin{aligned} &\frac{d\sigma_*}{dt} = -A_-\left(\sigma_* - \frac{\gamma}{\alpha_-}\right) - g, \quad \sigma_*(0) = -p_0, \quad 0 < t < t_B \\ &\frac{d\sigma_*}{dt} = -A_+\left(\sigma_* - \frac{\gamma}{\alpha_+}\right) - g, \quad \sigma_*(t_B) = \sigma_B, \quad t_B < t; \quad A_\pm = \frac{\alpha_\pm^2}{2\tau\beta} \end{aligned} \tag{3.8}$$

the solution of which can be written in the explicit form

$$\sigma_*(t)-\sigma_0 = \left(\sigma_\infty-\sigma_0+\frac{\alpha_+-\alpha_-}{\alpha_-}\sigma_A\right)\left[1-\exp\left(-\frac{\alpha_-^2}{\alpha_+^2}\frac{gt}{(\sigma_A-\sigma_\infty)}\right)\right],\quad \sigma_0\le\sigma_*(t)\le\sigma_B$$
$$\sigma_*(t)-\sigma_B = (\sigma_\infty-\sigma_B)\left[1-\exp\left(-\frac{g(t-t_B)}{\sigma_A-\sigma_\infty}\right)\right],\quad \sigma_B\le\sigma_*(t)<\sigma_\infty \tag{3.9}$$

The second of relations (3.9) shows that, for small relaxation times $\tau \ll 1$, the stress σ_* tends to the asymptotic value

$$\sigma_\infty = \frac{\gamma}{\alpha_+}-\frac{2\tau\beta g}{\alpha_+^2} = \sigma_A-\frac{2\tau\beta g}{\alpha_+^2} = \sigma_A-\frac{2\tau g}{1-b_+^2} \tag{3.10}$$

The value of σ_∞ exceeds the threshold $\sigma_A=\gamma/\alpha_+$ at which a damage begins. The appearance of the quantity σ_∞ is due to interference of the damage kinetics and the initial compression of the material. This effect does not occur in the case of the wave propagation in elastoviscoplastic bodies[14] and damaged materials without initial stresses,[15] where the asymptotic value is exactly equal to the threshold stress σ_A.

It can be seen from the solution (3.9) that the compressive stresses decay exponentially during the propagation of the wave. Since $\alpha_-^2/\alpha_+^2>1$, the fall in the amplitude of the bulk failure wave is more intense compared with the shear failure wave. The rate of change is proportional to the ratio of the velocity of free fall $g(t-t_B)$ to the width of the kinetic interval

$$\Delta_{\rm kin}\equiv\sigma_A-\sigma_\infty = 2\tau g/(1-b_+^2)$$

The dependence $\sigma_*(t)/\sigma_A$ of the amplitude of the head wave when $\sigma_0=10\sigma_A$ for a material with the parameters

$$\lambda=\mu=0.33,\ \alpha_p^+ = 0.3,\ \alpha_p^- = -0.1,\ \alpha_s = 1.1,\ \gamma = 0.01,\ \beta = 1.0,\ e_B = 3e_A \tag{3.11}$$

are shown for the three relaxation times $\tau=0.01$, 0.03, 0.05 (curves *1–3*) in Fig. 4, in which the kink in the curves, corresponding to the instant

$$t_B = -\frac{\alpha_+^2}{\alpha_-^2}\frac{(\sigma_A-\sigma_\infty)}{g}\ln\left[1-\frac{\sigma_B-\sigma_0}{\alpha_+\sigma_A/\alpha_--\sigma_0-2\tau g/(1-b_-^2)}\right] \tag{3.12}$$

when the transition from bulk to shear failure occurs, is clearly visible.

At a moderate pressure $\sigma_B<\sigma_0<\sigma_A$ the damage also starts to build up when $t=0$. In this case, shear failure occurs. The evolution of the amplitude of the head wave is given by the relation

$$\sigma_*(t)-\sigma_0 = (\sigma_\infty-\sigma_0)[1-\exp(-gt/(\sigma_A-\sigma_\infty))] \tag{3.13}$$

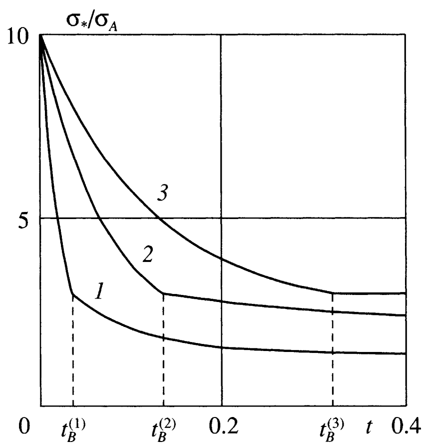

Fig. 4.

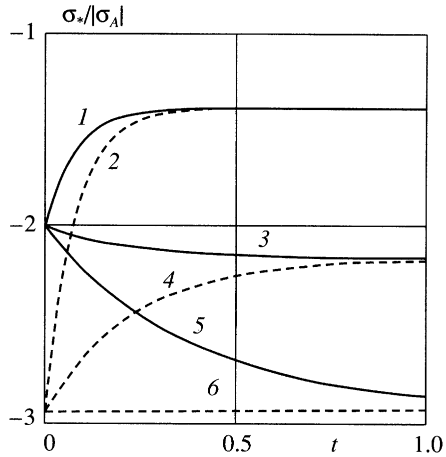

Fig. 5.

If the stress σ_0 applied to the boundary is in the range $\sigma_B < \sigma_0 < \sigma_\infty$, that is, outside the kinetic interval, then the coefficient $\sigma_\infty - \sigma_0$ in equality (3.13) is positive. The amplitude $|\sigma_*(t)|$ of the stress in the wave front decreases monotonically from $|\sigma_0|$ to the asymptotic value $|\sigma_\infty|$, given by expression (3.10).

If, however, the stress σ_0 belongs to the kinetic interval, that is, $\sigma_\infty < \sigma_0 < \sigma_A$, then the head wave propagation is accompanied by an increase in its amplitude $|\sigma_*(t)|$ to a value of σ_∞.

The relation $\sigma_*(t)/|\sigma_A|$ in the case of a moderate action when $\sigma_0 = 2\sigma_A$ (curves *1*, *3*, *5*) and $\sigma_0 = 2.9\sigma_A$ (curves *2*, *4*, *6*) is shown in Fig. 5 for the material parameters (3.11) when $\tau = 0.01$ (curves *1* and *2*), $\tau = 0.03$ (curves *3* and *4*) and $\tau = 0.05$ (curves *5* and *6*). It can be seen that, in the case of the specified values of the applied pressure σ_0, the amplitude of the head wave behaves differently depending on the relaxation time: it can decrease (curves *1*, *2* and *4*), increase (curves *3* and *5*) or remain constant (curve *6*). An increase in the amplitude is an effect which is directly associated with the initial stresses in the material.

The third case, which is the most interesting, occurs when the stress σ_0, applied to the boundary $x = 0$ is small, when $z_+(\sigma_0, 0) \equiv \alpha_+\sigma_0 - \gamma \le 0$. In the initial section of the propagation of the head wave there is no damage since the right-hand side of the kinetic equation is equal to zero. The material behaves in an elastic manner. The solution of problem (3.4)–(3.6) is piecewise-constant

$$\begin{aligned} &\sigma(x,t) = \sigma_0 - gx, \quad \upsilon(x,t) = -\sigma_0, \quad \omega(x,t) = 0, \quad 0 \le x \le t, \quad t > 0 \\ &\sigma(x,t) = -gx, \quad \upsilon(x,t) = 0, \quad \omega(x,t) = 0, \quad t < x, \quad t > 0 \end{aligned} \tag{3.14}$$

The maximum compression $\sigma(1,1) = \sigma_0 - gx$ occurs at the bottom of the layer $x = 1$ when $t = 1$. The material in this section remains undamaged subject to the condition

$$\sigma_0 \ge \sigma_0^e \equiv g + \gamma/\alpha_+$$

If, however, the load $|\sigma_0|$ applied to the boundary is larger than $|\sigma_0^e|$, it follows from relations (3.14) that the stress in the head wave reaches the threshold $\sigma_A = \gamma/\alpha_+$ at the point x_A:

$$\sigma_0 = gx_A + \gamma/\alpha_+ \tag{3.15}$$

after which the damage starts to build up, that is, the elastic precoursor transforms a shear failure wave. The stress $\sigma_*(t)$, which at the instant of time $t_A = x_A$ is equal to the sum of the applied and initial stresses, is given by the relation

$$\sigma_*(t) - \sigma_A = -(\sigma_A - \sigma_\infty)[1 - \exp(-g(t - t_A)/(\sigma_A - \sigma_\infty))], \quad t > t_A \tag{3.16}$$

The solution, when $\sigma_0 = 0.3\sigma_A$ (curves *1* and *2*) and $\sigma_0 = 0.9\sigma_A$ (curves *3, 4,* and *5*), is shown in Fig. 6 for a material with the parameters (3.11) and with a relaxation time $\tau = 0.001$ (curve *5*), $\tau = 0.01$ (curves *2* and *4*) and $\tau = 0.02$ (curves *1* and *3*). The solution which has been constructed suggests that, at small initial loads, bulk failure in the head shock wave does not occur and there is shear failure only. Unlike the derivative $d\sigma_*/dt = -g$ in the interval $t \le t_A$ at instants

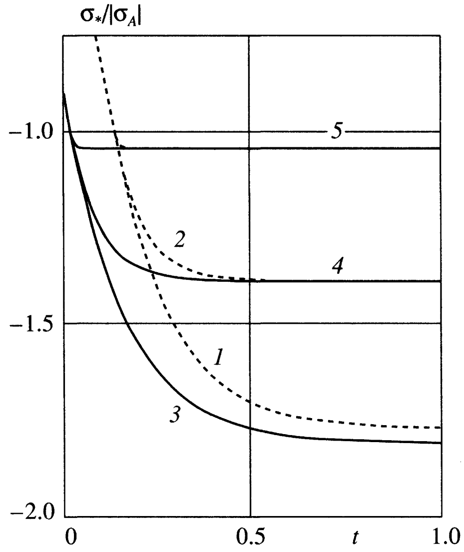

Fig. 6.

of time $t > t_A$, the derivative

$$\frac{d\sigma_*}{dt} = -g + A|\alpha_+|\frac{\alpha_+\sigma_* - \gamma}{2\tau} > -g$$

that is, the negative slope of the curve $\sigma_*(t)$, decreases. This means that prestressed material is unloaded in the neighbourhood of the head wave $x = t$, while remaining compressed below the threshold σ_A.

4. Macrofailure of a porous layer

The condition for rheological instability (2.9) enables us to determine how the macroscopic failure wave, corresponding to the points at which the strength criteria are satisfied, moves along a porous layer. It should be noted here that the system of equations (3.4), which is a system with a linear differential part with the velocities of the characteristics $c = \pm 1$, is not degenerate. Violation of the Hadamard condition occurs due to the other components of the strain tensor which differ from the e_{11} component of uniaxial strain. This means that macroscopic failure within the framework of the model being considered is a particularly multidimensional process which manifests itself in the form of strongly increasing small perturbations of the strain tensor.

As shown in Section 2, rheological instability of a material in a state of uniaxial strain sets in for a value of the parameter $\xi \equiv \alpha_s\omega/J(e)$ equal to $\xi^* = 2\mu$. This condition, as will be seen from the subsequent discussion, also determines the law of motion of the macrofailure wave.

In a shock wave, the equation of which in the characteristic variables

$$\zeta = (x + t)/2, \quad \eta = (t - x)/2$$

has the form $\eta = 0$, the damage $\omega(\zeta, 0) = 0$. The stress is given by relations (3.9), (3.13) and (3.16). The derivatives $\partial v/\partial\eta$, $\partial\sigma/\partial\eta$, $\partial\omega/\partial\eta$ are required in order to construct the solution behind the wave when $\eta > 0$. Taking account of the formulae

$$t = \zeta + \eta, \quad \partial\omega/\partial t|_{\zeta = \mathrm{const}} = \partial\omega/\partial\eta$$

from relations (3.7), we obtain

$$\frac{\partial\omega}{\partial\eta} = \frac{1}{\tau}z_\pm(\zeta, \eta), \quad z_\pm \equiv \frac{\alpha_\pm}{\beta}\left(\sigma - \frac{\gamma}{\alpha_\pm}\right) - b_\pm^2\omega, \quad b_\pm^2 = 1 - \frac{\alpha_\pm^2}{\beta}$$

From this, we find that, in the neighbourhood of the characteristic $\eta = 0$,

$$\omega(\zeta, \eta) = \frac{\eta}{\tau} z_{\pm}^*(\zeta), \quad z_{\pm}^*(\zeta) \equiv z_{\pm}(\zeta, 0) = \frac{\alpha_{\pm}\sigma_*(\zeta) - \gamma}{\beta} \tag{4.1}$$

Here, $z_{\pm}^*(\zeta)$ is a non-negative function with a discontinuity when $t = t_B$ as a consequence of the jump in $\alpha_{\pm}$.

In a small neighbourhood of the characteristic $\eta = 0$, the derivative $\partial\sigma/\partial\eta$ satisfies the first-order equation with the initial condition when $\zeta = 0$

$$\frac{dY_{\pm}}{d\zeta} + A_{\pm}Y_{\pm} = \frac{\alpha_{\pm}b_{\pm}^2}{2\tau^2} z_{\pm}^*(\zeta) - \frac{\alpha_{\pm}}{2\tau}\frac{dz_{\pm}^*(\zeta)}{d\zeta}, \quad Y_{\pm}(\zeta) = \left.\frac{\partial\sigma}{\partial\eta}\right|_0, \quad A_{\pm} = \frac{\alpha_{\pm}^2}{2\tau\beta} \tag{4.2}$$

$$Y_{\pm}(0) = g + A_{\pm}\left(\sigma_0 - \frac{\gamma}{\alpha_{\pm}}\right) = g + \frac{\alpha_+}{2\tau} z_{\pm}^*(0) \tag{4.3}$$

The plus sign corresponds to the points of ζ at which $\sigma_B < \sigma_*(\zeta) < \sigma_A$ and the minus sign to the points at which $\sigma_*(\zeta) < \sigma_B$. The solution of the Cauchy problem (4.2) and (4.3) has the form: for a moderate compression $\sigma_B < \sigma_0 < \sigma_A$

$$Y_+(\zeta) = \left(g + \frac{\alpha_+}{\tau} z_+^*(0)\right) e^{-A_+\zeta} - \frac{\alpha_+}{2\tau} z_+^*(\zeta) + B_+ e^{-A_+\zeta} \int_0^{\zeta} z_+^*(s) e^{A_+ s} ds \tag{4.4}$$

and, for a strong compression $\sigma_0 < \sigma_B$,

$$\begin{aligned} Y_-(\zeta) &= \left(g + \frac{\alpha_-}{\tau} z_-^*(0)\right) e^{-A_-\zeta} - \frac{\alpha_-}{2\tau} z_-^*(\zeta) + B_- e^{-A_-\zeta} \int_0^{\zeta} z_-^*(s) e^{A_- s} ds, \quad \zeta < \zeta_B \\ Y_+(\zeta) &= \left[Y_-(\zeta_B) + \frac{\alpha_+}{2\tau} z_+^*(\zeta_B)\right] e^{-A_+(\zeta - \zeta_B)} - \frac{\alpha_+}{2\tau} z_+^*(\zeta) + B_+ e^{-A_+\zeta} \int_{\zeta_B}^{\zeta} z_+^*(s) e^{A_+ s} ds, \quad \zeta > \zeta_B \end{aligned} \tag{4.5}$$

Here,

$$B_{\pm} = \frac{\alpha_{\pm}}{\tau^2}\left(1 - \frac{\alpha_{\pm}^2}{2\beta}\right) = \frac{\alpha_{\pm}(1 + b_{\pm}^2)}{4\tau^2}$$

and $\zeta_B = t_B$ is the instant of transition from bulk to shear failure, which is determined by formula (3.12).

To prove this, we differentiate the equation along the characteristic $\eta = \text{const}$ with respect to ζ.

$$\frac{\partial(\upsilon + \sigma)}{\partial\eta} = g - \frac{\alpha_{\pm}}{\tau} z_{\pm}(\zeta, \eta)$$

As a result, on the line $\eta = 0$, we obtain

$$\frac{\partial}{\partial\zeta}\left.\frac{\partial\upsilon}{\partial\eta}\right|_0 + \frac{\partial}{\partial\zeta}\left.\frac{\partial\sigma}{\partial\eta}\right|_0 = -\frac{\alpha_{\pm}}{\tau}\frac{dz_{\pm}^*(\zeta)}{d\zeta} \tag{4.6}$$

Another relation along the characteristic $\zeta = \text{const}$ has the form

$$\frac{\partial(\upsilon - \sigma)}{\partial\zeta} = g + \frac{\alpha_{\pm}}{\tau} z_{\pm}(\zeta, \eta)$$

Linearizing this equation in the neighbourhood of $\eta = 0$, we obtain

$$\frac{\partial}{\partial\zeta}\left(\upsilon_*(\zeta) + \eta\left.\frac{\partial\upsilon}{\partial\eta}\right|_0\right) - \frac{\partial}{\partial\zeta}\left(\sigma_*(\zeta) + \eta\left.\frac{\partial\sigma}{\partial\eta}\right|_0\right) = g + \frac{\alpha_{\pm}}{\tau} z_{\pm}^*(\zeta) + \frac{\alpha_{\pm}}{\tau}\eta\left.\frac{\partial z_{\pm}}{\partial\eta}\right|_0$$

Taking account of the relation

$$\frac{\partial\upsilon_*(\zeta)}{\partial\zeta} - \frac{\partial\sigma_*(\zeta)}{\partial\zeta} = g + \frac{\alpha_\pm}{\tau}z_\pm^*(\zeta)$$

$$\left.\frac{\partial z_\pm}{\partial\eta}\right|_0 = \frac{\alpha_\pm}{\beta}\left.\frac{\partial\sigma}{\partial\eta}\right|_0 - \left(1-\frac{\alpha_\pm^2}{\beta}\right)\left.\frac{\partial\omega}{\partial\eta}\right|_0 = \frac{\alpha_\pm}{\beta}\left.\frac{\partial\sigma}{\partial\eta}\right|_0 - \frac{1}{\tau}\left(1-\frac{\alpha_\pm^2}{\beta}\right)z_\pm^*(\zeta)$$

we find

$$\frac{\partial}{\partial\zeta}\left.\frac{\partial\upsilon}{\partial\eta}\right|_0 - \frac{\partial}{\partial\zeta}\left.\frac{\partial\sigma}{\partial\eta}\right|_0 = \frac{\alpha_\pm^2}{\tau\beta}\left.\frac{\partial\sigma}{\partial\eta}\right|_0 - \frac{\alpha_\pm}{\tau^2}\left(1-\frac{\alpha_\pm^2}{\beta}\right)z_\pm^*(\zeta) \tag{4.7}$$

Subtracting Eq. (4.7) from Eq. (4.6), we arrive at Eq. (4.2).

The initial condition (4.3) follows from the equality $\sigma(\zeta, \zeta) = \sigma_0$ if account is taken of the fact that the derivative $2\partial\sigma/\partial t = \partial\sigma/\partial\zeta + \partial\sigma/\partial\eta = 0$ on the boundary $x = \xi - \eta = 0$ and formula (3.8) for the derivatives of the stresses in the head wave are equal to zero. The correctness of the representations (4.4) and (4.5) is easily established by direct verification.

It is now possible to formulate the equation for macrofailure. Using the instability condition (2.9), the relation $J = -\sqrt{2/3}(\sigma + \alpha_\pm\omega)$, expression (4.1) and the expansion

$$\sigma(\zeta, \eta) = \sigma_*(\zeta) + \eta\partial\sigma/\partial\eta|_0 + O(\eta^2)$$

where the derivative is given by formulae (4.4) and (4.5), for the macrofailure wave we obtain

$$x(t) = t - 2|\sigma_*(t)|/F_\pm(t), \quad F_\pm(t) = \tau^{-1}(\sqrt{3/2}\alpha_s/\xi^* - |\alpha_\pm|)z_\pm^*(t) + Y_\pm(t) \tag{4.8}$$

The amplitude of the head wave $\sigma_*(t)$ is given by one of the formulae (3.9), (3.13), (3.16), the positive quantity $z_\pm^*(\zeta)$ is given by expression (4.1) and the function $Y_\pm(\zeta) \equiv \partial\sigma(\zeta, 0)/\partial\eta$ is given by relations (4.4) and (4.5) where $\zeta = (x + t)/2$.

The coordinate of the macrofailure wave $\eta_\pm^*(\zeta) \equiv (t - x)/2 > 0$. This is possible when $F_\pm(\zeta) > 0$. Since the parameter ξ^* only appears in the first term of the function $F_\pm(\zeta)$, the value of $F_\pm(\zeta)$ is a maximum in the case of the smallest value of $\xi^* = 2\mu$, which corresponds to the fastest macrofailure wave. The value $\xi^* = 2\mu$ either gives degeneracy with respect to the velocity ρc_3^2, with an arbitrary orientation of the surface of a strain localization or degeneracy with respect to the velocity ρc_2^2 to which the first form of rheological instability corresponds. This means that, within the framework of the problem of uniaxial strain, it is impossible to determine the form in which rheological instability (the character of the macrofailure) manifests itself. A numerical solution of a multidimensional problem is required for this.

The macrofailure wave fronts $x = x_c(t, \sigma_0)$ are shown on the left in Fig. 7 for a material with the parameters (3.11) and $g = 0.1$ and $\tau = 0.02$ for three moderate loading values $\sigma_B < \sigma_0 < \sigma_A$. The dashed curve corresponds to the head wave $x = t$ and curves *1–3* correspond to the lines $x = x_c(t, \sigma_0)$ for $\sigma_0/\sigma_A = 2.8, 2.4, 2.3$.

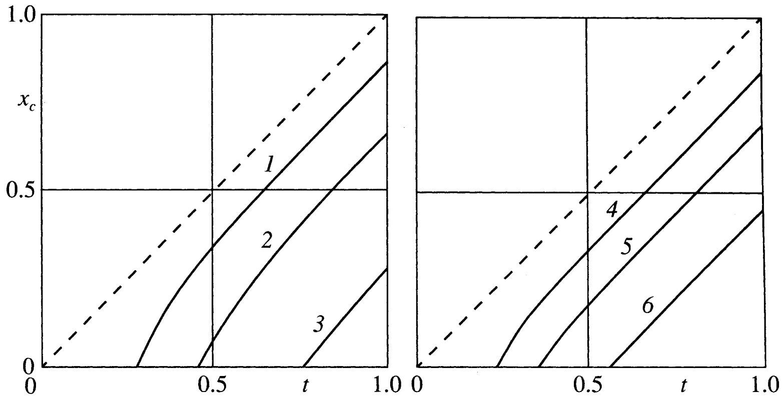

Fig. 7.

The non-trivial effect of the relaxation time on the motion of the macrofailure wave should be noted. The relations $x = x_c(t, \sigma_0)$ are shown on the right of Fig. 7 in the case of a relaxation time $\tau = 0.01$ for $\sigma_0/\sigma_A = 2.66$, 2.70, 2.8 (curves *4–6*). In this case, a macrofailure wave arises for a stress $\sigma_0/\sigma_A > 2.6$, while, when $\tau = 0.02$, macrofailure starts for a load $\sigma_0/\sigma_A > 2.25$. This means that the resistance of the body increases as the relaxation time τ decreases. An increase in the shear modulus also promotes an increase in the resistance of the porous layer. Shear macrofailure begins for the above mentioned values of the parameters when $\sigma_0/\sigma_A > 2.66$ for $\mu = 0.45$, $\lambda = 0.10$, $\tau = 0.01$.

In the case of intensive dynamic loading $|\sigma_0| > |\sigma_B|$, the function $x = x_c(t, \sigma_0)$ depends only slightly on the applied pressure and practically coincides with the curve $x = x_c(t, \sigma_0)$, corresponding to the start of the bulk failure of the material. This is due to rapid attenuation of the stress wave in the neighbourhood of the boundary $x = 0$ to a value σ_∞, the meaning of which has been discussed in detail in Section 3.

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